Nature of Integer Quantum Hall Transitions

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The mapping between the metal-insulator transition of the quantum Hall system and a superfluid-to-insulator transition is revisited based on a disordered anyon model. The one-parameter scaling of the superfluid-to-insulator transition is employed for the analysis of the scaling behavior of the integer quantum Hall transitions. The analysis reveals the direct transition from a quantum Hall plateau to the insulator in weak disorder limit, and a float-down transition for strong disorders. In either cases, the transition corresponds to a non-chirality superfluid-to-insulator transition, with the longitudinal and transverse quantum Hall conductivities following a semi-circle relation.

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The problem of metal-insulator transition in quantum Hall systems has attracted much attention [1-4] since the discovery of integer quantum Hall (IQH) effect. As the scaling theory of localization [5] predicts localization of all non-interacting electrons in two dimensions, a natural challenge has been to understand the extended states at the center of disorder-broadened Laudau bands in the IQH system. Laughlin [1] and Khmelnitskii [2] proposed a quasi-classical theory on the float-up of the extended states near the band edge. Based on the float-up scenario, Pruisken [3] concluded that there exists two sets of fixed point in the renormalization group phase diagram for the IQH system. One set of stable fixed point is located at $\sigma_{xx}^H = 0$ and $\sigma_{xy}^H = n$ (in the unit of e^2/h), corresponding to the Hall plateau. The other set of unstable fixed points is located at σ_{xx}^{H*} (independent of n) and $\sigma_{xy}^{H*} = n \pm \frac{1}{2}$, corresponding to the center of each Laudau band, which signals a localization-delocalization phase transition. A renormalization group flow diagram showing two-parameter scaling for the IQH system was established; the Hall conductance appears as a second coupling constant, in addition to the dissipative conductance. The sets of fixed points are subsequently employed to the study of the global phase diagram of the quantum Hall effect [4].

Kivelson, Lee and Zhang [4] (KLZ) proposed a global phase diagram for the quantum Hall system based on a mapping of the corresponding states of the metal-to-insulator transition to ones of a superfluid-to-insulator transition. The theory predicted the critical conductance $\sigma_{xx}^{H*} = \frac{1}{2}$ that is supported by subsequent simulation studies for $n = 1 \rightarrow 0$ transition [6]. The global phase diagram was then constructed using symmetry transformations of the float-up picture, including the particle-hole, flux attachment, and the Laudau level addition transformations. By construction, the global phase diagram allows only nearest-neighbor IQH plateau transitions. However, the direct transitions from high IQH plateau ($n > 1$) to insulator have been observed by both experiments [7-8] and computer simulations [9-13]. It was argued that the incapability of predicting the direct transitions in the global phase diagram is attributed to the lattice effects that are not present in the continuum models.

In this Letter, I revisit the mapping between the metal-to-insulator transitions and the superfluid-to-insulator transition based on a disordered anyon model [14]. The resultant mapping relations are essentially the same as those obtained by KLZ in the weak disorder limit. However, closer scrutiny of the self-duality relations without invoking the symmetry transformations *a priori*, reveals the existence of direct transitions form high quantum Hall plateaus to the insulator. This finding and the associated predictions based on one-parameter scaling flow are remarkably in agreement with the numerical simulations [9-13]; thereby providing important information on the microscopic origin of the direct transitions. Subsequently, we consider the strong disorder limit where the float-down transitions become appropriate. A distinct feature of our analysis is the realization that a superfluid-to-insulator transition with non-chirality constraint, maps to a quantum Hall critical region where the longitudinal and transverse quantum Hall conductivities satisfy a semi-circle relation. The semi-circle relation was proposed by Ruzin and co-workers [15] based on a phenomenological two-fluid theory.

Statistical mapping.$-$The disordered anyon model is described by the Hamiltonian [14,16]

$$\hat{H} = \sum_i \left\{ \frac{1}{2m} \left(\mathbf{P}_i - \hat{\mathbf{A}}(\mathbf{r}_i) \right)^2 + V(\mathbf{r}_i) \right\}, \qquad (1)$$

where $V(\mathbf{r}_i)$ is short-range correlated random potential, and $\hat{\mathbf{A}}(\mathbf{r}_i)$ is a statistical field operator given by

$$\hat{\mathbf{A}}(\mathbf{r}_i) \equiv \frac{\hbar}{n} \sum_j \frac{\hat{\mathbf{z}} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \qquad (2)$$

with $\hat{\mathbf{z}}$ a unit vector normal to the plane, and n the so-called statistics parameter ($n = 1$ for bosons; $n = 2$ for semions; and $n = \infty$ for fermions) characterizing

the specific form of the fractional statistics. The system is equivalent to spinless fermions interacting through Chern-Simons gauge field and disordered impurity scattering. It is worth noting that disordered-fermion system stands for the $n = \infty$ limit of the model.

We first consider the average statistical field that yields a fictitious magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The resulting mean-field Hamiltonian

$$\hat{H}_0 = \sum_i \frac{1}{2m}(\mathbf{P}_i - \bar{\mathbf{A}}_i)^2 + V(\mathbf{r}_i), \tag{3}$$

describes a collection of spinless particles moving in a strong magnetic field $\mathbf{B}$ with n filled Laudau bands and the random potential V. The equivalence between the IQH system and the mean-field solution of the disordered anyon system is crucial in the statistical mapping.

The effect of statistical-field fluctuations around the mean-field solution, $\hat{H} - \hat{H}_0$, can be evaluated by a consistent $1/n$-expansion scheme [17]. In the absence of the random potential, the leading-$1/n$-order results are the same as those obtained in the calculation of the random-phase-approximation (RPA) response functions [14,16]. Based on a correlated RPA calculation [17], we have shown that the next-leading-order corrections to the RPA results have no effect in the long-wavelength limit. As a consequence, the longitudinal and transverse conductivities, σ_{xx} and σ_{xy}, can be evaluated from the leading-$1/n$-order current-current correlation functions at $T = 0$, and are related to those for IQH systems, σ^H_{xx} and σ^H_{xy}, by expressions [14,17]:

$$\sigma_{xx} = n^2 \frac{\sigma^H_{xx}}{(\sigma^H_{xx})^2 + (\sigma^H_{xy} - n)^2}, \tag{4}$$

$$\sigma_{xy} = \sigma_R + n^2 \frac{\sigma^H_{xy} - n}{(\sigma^H_{xx})^2 + (\sigma^H_{xy} - n)^2}. \tag{5}$$

Here $\sigma_R = n$ in the absence of disorders. This set of equations is equivalent to that obtained by KLZ [4] via a transformation $n \to 1/n$. In fact, they become identical if one changes the unit of conductivities in Eqs. (4-5) from e^2/h to e^{*2}/h where $e^* = ne$.

The mapping of conductivities between the disordered anyon system and the IQH system is self-dual in that exchanging $(\sigma^H_{xx}, \sigma^H_{xy})$ with $(\sigma_{xx}, \sigma_{xy})$, the function form of Eqs. (4-5) remains invariant. The self-duality is the ramification of the particle-vortex duality in connection to the Chern-Simons gauge field.

It is worth pointing out that the effect of disorder plays an important role in the IQH systems. As such, it is necessary to evaluate the effect of disorder on the mapping relations. In the weak disorder limit, the self-duality equations was argued to be intact [14]. The weak-disorder limit refers to the case that there exists IQH effect for n Landau bands. In this case, the contribution to σ_R for the anyon model can be calculated using the quasi-classical theory [1-2].

Considerable insight of the scaling behavior of the quantum Hall transitions can be gained from the mapping [4,14]. To this end, we first construct the corresponding states between the two systems. From Eqs. (4-5), $(\sigma^H_{xx}, \sigma^H_{xy}) = (0, 0)$ maps to $(\sigma_{xx}, \sigma_{xy}) = (0, 0)$, corresponding to an insulating state. The "localization" fixed points associated with the plateau solutions in the IQH system correspond to two types of states in the disordered anyon system. $(\sigma^H_{xx}, \sigma^H_{xy}) = (0, m)$ $(m \neq n)$ maps to a quantum Hall conductor, and $(\sigma^H_{xx}, \sigma^H_{xy}) = (0, n)$ was shown to map to $(\sigma_{xx}, \sigma_{xy}) = (\infty, n)$, corresponding to an anyon-type superfluid state [14], characterized by a spontaneous breaking of parity and time-reversal invariance. Based on the concept of corresponding states, KLZ [4] constructed the global phase diagram using symmetry transformations.

Direct transition.—While the corresponding state for the insulator is unique, the corresponding state for the superfluid state is not. As a matter of fact, the corresponding state for $(\sigma^H_{xx}, \sigma^H_{xy}) = (0, n)$ depends on the approaching path. For our purpose, we are motivated to search for a critical flow without breaking the parity and time-reversal symmetries, characterized by $\sigma_{xy,c} = 0$. In the context of two-parameter renormalization flow, this amounts to finding the critical regions that follow a one-parameter (here it stands for $\sigma_{xx,c}$) scaling. From Eq. (5), one has

$$\sigma_{xy,c} = n^2 \frac{\sigma^H_{xy,c} - n}{(\sigma^H_{xx,c})^2 + (\sigma^H_{xy,c} - n)^2} + n = 0. \tag{6}$$

The solution of the above equation leads to a flow line where the longitudinal and transverse quantum Hall conductivities follow a semi-circle relation:

$$(\sigma^H_{xx,c})^2 + (\sigma^H_{xy,c} - \frac{n}{2})^2 = (\frac{n}{2})^2. \tag{7}$$

Summarized in Table I are the corresponding states in concord with the one-parameter scaling flow. It is interesting to observe that an anyon superfluid state without breaking the parity and time-reversal symmetries can be reached if the corresponding quantum Hall conductivities following a semi-circle relation in the critical region. From a general two-parameter scaling point of view, it is expected a general points at the $\sigma^H_{xx} - \sigma^H_{xy}$ plane will flow into the semi-circle critical regions. There exists three fixed points in the critical region: two stable ones at $(\sigma^H_{xx}, \sigma^H_{xy}) = (0, 0)$ and $(\infty, 0)$, corresponding to the insulator and superfluid states, respectively. The unstable one is located at $(\sigma^{H*}_{xx}, \sigma^{H*}_{xy}) = (n/2, n/2)$, corresponding a quantum normal critical point for the non-chirality superfluid-to-insulation transition point with a critical normal conductance σ_R.

It is intriguing to note that the direct metal-insulator transition is exactly what Sheng and Weng [13] have identified based on their tight-binding lattice model study of

TABLE I. The corresponding states between the IQH system and disordered anyon model phase diagrams, following the semi-circle relationship for conductivities of quantum Hall systems that corresponds to non-chirality superfluid-to-insulator flow (with a universal critical conductance σ_R) for the disordered anyons.

σ_R	$(\sigma_{xx}^H, \sigma_{xy}^H)$	State	$(\sigma_{xx}, \sigma_{xy})$	State
n	$(0,0)$	Insulating	$(0,0)$	Insulating
	$(0,n)$	IQH plateau	$(\infty,0)$	Superfluid
	$(\frac{n}{2},\frac{n}{2})$	Critical; Normal	$(n,0)$	Critical; Normal
n^2	$(0,n-1)$	IQH plateau	$(0,0)$	Insulating
	$(0,n)$	IQH plateau	$(\infty,0)$	Superfluid
	$(\frac{1}{2},n-\frac{1}{2})$	Critical; Normal	$(n^2,0)$	Critical; Normal

the IQH effect. Apart from the case of $n=1$, it is different from the nearest-neighbor plateau transitions. Our analysis shows that for this universality class, the critical fixed points, $(\sigma_{xx}^{H*}, \sigma_{xy}^{H*}) = (n/2, n/2)$, can be derived from the symmetry considerations of the duality transformation.

An important implication of the above mapping analysis is that the longitudinal conductance of the IQH system follows a scaling form (using the self-dual transformation Eqs. (4-5) and the semi-circle relation)

$$\frac{\sigma_{xx}^H}{\sigma_{xx}^{H*}} = \frac{2(\sigma_{xx}/\sigma_R)}{1+(\sigma_{xx}/\sigma_R)^2}. \tag{8}$$

Remarkably, this is exactly the form that has been found empirically from numerical simulations [10,13].

It is gratifying to show that the direct transitions are intimately connected to the particle-vortex duality intrinsic in the self-duality mapping. It becomes clear that the symmetry transformations used in the global phase diagram [4], however, are in general not in conformity to the self-duality. The self-duality between the particle and vortex yields non-trivial coupling and/or merging of the edge states.

Float-down transition.−With the increase of disorder, the IQH effect for the high Laudau band disappears one after another [11]. As a result, the particle-vortex duality is broken. Thus we need to consider the effect of disorder on the change of mapping relations. From the correlated RPA equations [17], it is worth noting that for averaging over the disorder, the longitudinal and transverse conductivities appearing in Eqs (4-5) are treated in the equal footing. The only part that needs to be reevaluated is the disconnected part that contributes to σ_R. In principle, the contribution depends on the strength and the specific form of the disorder. However, one can consider the strong disorder case when all the extended states of the anyons are projected onto the lowest Laudau level for composite fermions. In this limit, the vortices are pinned by disorder and bidding together as composite fermions, with charge ne. One can argue that the contribution of a charge ne composite fermion with flux quantization h/ne is $\sigma_R = n^2$. This is equivalent to set σ_R equal to 1 in the unit of e^{*2}/h ($e^* = ne$), the value for the lowest Landau level.

With $\sigma_R = n^2$, it is readily observable from Eqs. (4-5) that the delocalization point is located at $\sigma_{xx}^{H*} = \frac{1}{2}$ and $\sigma_{xy}^{H*} = n - \frac{1}{2}$, in concord with the float-up fixed points. Also listed in Table I are the corresponding states for this case. In contrast to the direct transition scenario where there exits a particle-vortex duality, here $(\sigma_{xx}^H, \sigma_{xy}^H) = (0, n-1)$ maps to the insulating state $(\sigma_{xx}, \sigma_{xy}) = (0,0)$. The float-down transitions can be viewed as follows: $(\sigma_{xx}^H, \sigma_{xy}^H) = (0,n)$ maps to a superfluid state for n-anyons. Through a metal-to-insulator transition following a semi-circle law

$$(\sigma_{xx,c}^H)^2 + (\sigma_{xy,c}^H - n + \frac{1}{2})^2 = (\frac{1}{2})^2, \tag{9}$$

which corresponds to a superfluid-to-insulator transition for $n-$anyons, it reaches $(\sigma_{xx}^H, \sigma_{xy}^H) = (0, n-1)$ that is an insulating state for $n-$anyons as well as a superfluid state for $(n-1)-$anyons. This leads to successive multi-step transitions, in agreement with the scenario provided by the global phase diagram for float-down transitions [4].

It is interesting to note that the projection of extended states onto the lowest Laudau band for composite fermions refers to a situation appropriate for the fractional quantum Hall systems. The symmetry transformations employed in the global phase diagram are valid for fractional quantum Hall transitions. Therefore, the float-down transition can be realized for system of composite fermions. In general, both direct and float-down transitions can happen in the IQH phase diagram. A detailed determination of the complicated phase boundaries requires the reference of numerical simulations [9-13]. It is attempting to argue that there exists two critical magnetic fields, B_{c1} and B_{c2}. When B_{c1} is reached from low-B side, the phase diagram shows direct transition, following a scaling behavior as given by Eq. (8) [10]. In the region $B_{c1} < B < B_{c2}$, there exist regions for float-down transitions, and the one-parameter scaling form of Eq. (8) is no longer valid. A detailed analysis of the IQH phase diagrams deserves further studies.

Semi-circle relation.−The above analysis can be extended to the general case of $n \to m$ plateau-plateau transitions. The critical region is defined by a semi-circle relation:

$$(\sigma_{xx,c}^H)^2 + (\sigma_{xy,c}^H - \frac{n+m}{2})^2 = (\frac{n-m}{2})^2. \tag{10}$$

Remarkably, this is exactly the form that Ruzin and coworkers [15] proposed based on phenomenological arguments. Our analysis shows that this critical region corresponds to a non-chirality superfluid-insulator transition governed by one parameter scaling. The success in extracting this phenomenological theory from the mapping

relations provides strong support for the correctness and consistency of our analysis.

Conclusions.−The revisit of the mapping relations between the superfluid-to-insulator transition and the metal-insulator transition of the IQH system reveals the existence of two types of universality classes in the IQH system: the direct transitions in the weak disorder limit, and the float-down transition in the strong disorder limit. Our analysis is based on the physics intuition that the one-parameter scaling of the non-chirality superfluid-to-insulator transition corresponds to critical regions for the quantum Hall system. The resulting predictions provides theoretical foundation for features observed in computer simulations based on the lattice model [9-13], such as the the existence of plateau-insulator direct transition universality class [11] and the universal scaling form of conductancei [10-11]. In addition, the analysis shed important light into the microscopic origin of the phenomenological semi-circle relation [15] for the longitudinal and transverse quantum Hall conductivities at the critical region.

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[1] R. B. Laughlin, Phys. Rev. Lett. **52**, 2304 (1984).

[2] D. E. Khmelnitskii, Phys. Lett. **106A**, 182 (1984).

[3] A. M. M. Pruisken, Phys. Rev. Lett. **61**, 1297 (1988).

[4] D. H. Lee, S. Kivelson, and S. C. Zhang, Phys. Rev. Lett. **67**, 3302 (1991); **68**, 2386 (1992); S. Kivelson, D. H. Lee, and S. C. Zhang, Phys. Rev. B **46**, 2223 (1992).

[5] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979); P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[6] B. Huckestein, Rev. Mod. Phys. **67**, 357 (1995); X. Wang, Q. Li, and C. M. Soukoulis Phys. Rev. B **58**, R3576 (1998).

[7] S. V. Kravchenko, W. Mason, J. E. Furneaux, V. M. Pudalov, Phys. Rev. Lett. **75**, 910 (1995).

[8] S. H. Song, D. Shahar, D. C. Tsui, Y. H. Xie, and D. Monroe, Phys. Rev. Lett. **78**, 2200 (1997).

[9] D. Z. Liu, X. C. Xie, and Q. Niu, Phys. Rev. Lett. **76**, 975 (1996).

[10] D. N. Sheng, Z. Y. Weng, and X. G. Wen, Phys. Rev. B **64**, 165317 (2001).

[11] D. N. Sheng and Z. Y. Weng, Phys. Rev. Lett. **78**, 318 (1997).

[12] Y. Hatsugai, K. Ishibashi, and Y. Morita, Phys. Rev. Lett. **83**, 2246 (1999).

[13] D. N. Sheng and Z. Y. Weng, Phys. Rev. Lett. **80**, 580 (1998).

[14] A. G. Aronov and A. D. Mirlin, Phys. Lett. **A152**, 371 (1991); X. Q. Wang, unpublished.

[15] I. Ruzin and S. Feng, Phys. Rev. Lett. **74**, 154 (1995); A. M. Dykhne and I. M. Ruzin, Phys. Rev. B **50**, 2369 (1994).

[16] A. L. Fetter, C. B. Hanna, and R. B. Laughlin, Phys. Rev. B **39**, 9679 (1989); Y. H. Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989).

[17] B. H. Liu, X. Q. Wang, and L. McLerran, Phys. Rev. B **43**, 13736 (1991).